# Waves in strongly magnetized relativistic plasmas: Generally covariant approach

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A dispersion relation for long waves in strongly magnetized multifluid plasma in a curved spacetime is derived in a covariant form. A generally covariant form for the ray equations is obtained. The results are applicable to ray propagation in relativistic plasmas in the vicinity of strongly gravitating (black holes) or rapidly rotating (pulsars) systems.

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Relativistic plasma in a superstrong magnetic field is a feature of a number of high-energy astrophysical systems including pulsars and black hole magnetospheres. Such systems also typically have rapid rotation and strong gravitational fields. In these cases, the use of a general relativistic approach is either mandatory (as near the black holes) or at least useful in the case in which the field pattern in the pulsar magnetosphere is assumed stationary in the rotating frame. The generation of plasma turbulence, the emission of radio waves, and the propagation of radio waves and other low-frequency disturbances through the magnetosphere are usually treated with no clear distinction made between the noninertial (corotating) frame and the inertial frames fixed to the star or to the observer. In this approach, it is implicitly assumed that wave solutions have the dependence $\exp(-i\omega t)$ in all frames, although this is, strictly speaking, incorrect in the observer's frame where the parameters of the ambient medium depend on time. The dispersion relation for each natural wave mode of the system is obtained by using an appropriate model for the waves. The long-wavelength (wavelength much larger than the particle gyroradius) low-frequency (frequency much smaller than the gyrofrequency) limit suffices for most purposes, and the usual approach is to use a kinetic equation in the plasma or pulsar frame [1–4]. The standard WKB approach uses the eikonal expansion where all perturbations $\propto\exp(iS/\eta)$, $\eta\sim\lambda/L\sim 1/\omega T$ being a small parameter ($\lambda$ is the typical wavelength, $L$ is the typical inhomogeneity scale, and $T$ is the typical time scale of nonstationarity). The lowest order gives the dispersion relation, which in the 3+1 method [5] is solved for the frequency $\omega=\omega(\mathbf{k},\mathbf{r},t)$, with the dependence on $\mathbf{r}$ and $t$ being due to inhomogeneity and time dependence of the system, both of which are necessarily present in the presence of rotation and a gravitational field. The four-dimensional dispersion relation in a local inertial frame can be written in the form $D(k,x)=0$, where $D$ is a Lorentz invariant and $k=(\omega/c,\mathbf{k})$, $x=(ct,\mathbf{r})$ are 4-vectors [6]. Here $k_a=\partial S/\partial x^a$. Further consideration of the wave propagation within the geometrical optics approach also requires careful treatment of the difference between the inertial nonrotating frame and the noninertial rotating frame. In the former, the time dependence of the background density and fields requires use of the complete set of the geometrical optics equation where $\omega=\omega(\mathbf{k},\mathbf{r},t)$. This was recognized by Barnard and Arons [7] but afterwards consistently ignored in the studies of ray propagation (see also Petrova and Lyubarskii [8] and references therein). In the pulsar frame, the parameters of the plasma depend on $\phi-\Omega t$, where $\phi$ is the azimuthal angle about the rotation axis and $\Omega$ is the angular velocity. A time derivative is then of the form $(\partial/\partial t)\sim(\Omega r)\nabla$, and the effects of time dependence may be significant. One way to treat this problem would be to consider it in the rotating frame where the background parameters are time-independent. However, this frame is noninertial and formally this needs to be taken into account through the metric tensor. The metric tensor can be approximated by its flat-space form only for $\Omega r\ll 1$ (here and hereafter the light velocity $c\equiv 1$ for convenience), and this factor varies from $\Omega r\sim 10^{-4}/P$ near the surface of a neutron star rotating with period $P$ ($10^{-3}\lesssim P\lesssim 1$s) to unity at the light cylinder.

In general, gravitating and/noninertial systems should be described within the covariant approach of general relativity. Derivation of the response tensor of the plasma is well-established in the special relativity (see, e.g., Melrose [6] and references therein) within the kinetic approach. It involves writing the linear response of the plasma in terms of the (Fourier-transformed) 4-current $j^a(k)$ in terms of the 4-potential, $A^a(k)$: $j^a(k)=\Pi^{ab}(k)A_b(k)$. Here indices span the range 0–3, with the metric tensor $g_{ab}=\mathrm{diag}(1,-1,-1,-1)$ in flat space time. Upon substituting the current into Maxwell equations, one arrives at the equation of the kind $D_{ab}A^b=0$, where $D_{ab}=g_{ab}k^2-k_ak_b-4\pi\Pi_{ab}$. The tensor $\Pi^{ab}(k)$ satisfiying $k_a\Pi^{ab}(k)=0=k_b\Pi^{ab}(k)$ [charge continuity $k_aj^a(k)=0$ and gauge invariance], and the determinant of $D_{ab}$ is identically zero. A covariant form, $D(k)=0$, of the dispersion equation is found by noting that the matrix of cofactors of $D_{ab}$ is of the form $D(k)k^ak^b$. An alternative way (used below) is to construct invariant combinations by projecting onto a set of independent vectors. This theory reproduces the conventional plasma dispersion theory in any specific inertial frame.

Generally relativistic analysis usually requires determination of wave properties in a frame that is not necessarily (locally) inertial and is less elaborate. Breuer and Ehlers [9] outlined the derivation of the dispersion relation in the simplest case of a cold electron plasma. However, no closed expression for the dispersion relation was provided that could be useful for applications, and too severe restrictions were imposed in the course of the derivation (such as the absence of ion current). Gedalin and Oiberman [10] analyzed magnetohydrodynamical waves in a curved space time. Elasässer and Popel [11] derived a local dispersion relation for a nonmagnetized plasma in the 3+1 form using a covariant approach. Generally covariant geometrical optics has been developed for the ray propagation in the vacuum (see, e.g., Stephani [12]) and in a formal way for gravitational and sound waves [13]. In the present paper, we present a concise derivation of the dispersion relation for a strongly magnetized multifluid plasma and a general relativistic formalism for the geometric optics treatment of the propagation of waves in an arbitrary medium, with a particular application to rapidly rotating systems.

We start with the decomposition of the Maxwell tensor. Let $U^a$ be a global unit timelike vector (4-velocity): $U^aU_a=1$. Then one can split $F^{ab}$ for the background system into the electric and magnetic fields as follows [10]:

$$E^a=F^{ab}U_b, \quad B^a=\epsilon^{abcd}F_{bc}U_d, \tag{1}$$

$$F^{ab}=(E^aU^b-E^bU^a)+\frac{1}{2}\epsilon^{abcd}(U_cB_d-U_dB_c), \tag{2}$$

where $\epsilon_{abcd}$ is the completely antisymmetric tensor, $\epsilon_{0123}=\sqrt{|g|}$. We call the plasma magnetized if there exists $U^a$ such that $E^a\equiv 0$. This condition may be fulfilled only in part of the space. It is worth mentioning that, in general, $U^a$ does not have to coincide with the plasma velocity (cf. Breuer and Ehlers [9] and Gedalin and Oiberman [10]). For convenience, we define the unit vector $b^a=B^a/\sqrt{-B^aB_a}$ with $b^ab_a=-1$.

Dispersion theory in a multifluid plasma hydrodynamics model can be treated in the following generally covariant manner. The equations for the plasma dynamics are

$$(n_su_s^a)_{;a}=0, \quad T^{ab}_{s;b}=q_su_{sb}F^{ab}, \tag{3}$$

$$\epsilon^{dabc}F_{ab,c}=0, \quad F^{ab}_{;b}=-4\pi j^a, \tag{4}$$

$$j^a=\sum_s q_sn_su_s^a, \tag{5}$$

where $s$ denotes plasma species and a semicolon stands for the covariant derivative. The energy-momentum tensor of an ideal fluid is $T^{ab}=(e+p)u^au^b-pg^{ab}$. The set (3)–(5) should be completed with the equation of state. In what follows, we assume the adiabatic state equation with $p=p(n)$ and $de=[(e+p)/n]dn$.

The above equations describe global plasma flow (wind, accretion flow) as well as waves. Let the waves be described in terms of perturbed quantities $\delta n$, $\delta u^a$, $\delta F^{ab}$. In what follows, we consider the perturbations in the WKB limit, that is, the wavelength is assumed to be much smaller than the typical length of the inhomogeneity, including variations of the metric tensor $g_{ab}$. Thus in the equations for perturbations, covariant derivatives can be replaced by ordinary partial derivatives.

Using the usual WKB technique, we assume that all variables $\propto\exp(iS/\eta)$ with $S_{,a}\equiv k_a$, $k^bk_{a,b}\ll k_a$, and the presence of the small parameter $\eta\ll 1$ ensures the usual eikonal expansion. Then the equations for the perturbations take the following form in the lowest order (we omit subscript $s$ for convenience):

$$k_a(\delta nu^a+n\delta u^a)=0, \tag{6}$$

$$ik_b\delta T^{ab}=q(\delta u_bF^{ab}+u_b\delta F^{ab}), \tag{7}$$

$$\delta T^{ab}=\left(\frac{e+p}{n}+c^2\right)\delta nu^au^b+(e+p)(u^a\delta u^b+u^b\delta u^a) -c^2\delta ng^{ab}, \tag{8}$$

where $c^2=dp/dn$ and $\delta u_a\cdot u^a=0$. It is worth noting that the transition to ordinary derivatives, that is, neglect of the affinity (Cristoffel symbols) in the lower order, is mandatory for the WKB approximation. The affinity enters in the higher-order equations for ray propagation (see below). The system (6)–(8) with the corresponding Maxwell equations for the field perturbation can be solved in the general case. Here we are interested in the strong magnetic-field approximation where one can formally set $B^2=-B_aB^a\to\infty$. In this case, the analysis simplifies greatly. One has $u_bF^{ab}=\delta u_bF^{ab}=0$ and it is easy to show that one can write

$$u^a=\gamma(U^a+vb^a), \quad \delta u^a=\delta u(vU^a+b^a), \tag{9}$$

where $v=u/\gamma$ and $\gamma^2=1+u^2$. Strong magnetization often means plasma anisotropy, $p_\perp\neq p_\parallel$, where $\perp$ and $\parallel$ refer to the magnetic-field direction. In pulsar magnetospheres, $p_\perp=0$. It can be shown [10] that in this case $T^{ab}=\epsilon u^au^b+p_\parallel b^ab^b$ and $c^2=dp_\parallel/dn$. Equation (6) immediately gives

$$\delta n=-\frac{n(Wv-K_\parallel)}{\gamma(W-K_\parallel v)}\delta u, \tag{10}$$

where we introduced the notation $W=k_aU^a$, $K_\parallel=-k_ab^a$. It is clear that $W$ and $K_\parallel$ are the frequency and parallel component of the wave vector (with respect to the magnetic field) in the locally inertial frame defined by the velocity $U^a$. Substituting Eq. (10) into Eq. (8) and further into Eq. (7), one gets after multiplication by $U_a$,

$$\delta u=-\frac{iq(W-K_\parallel v)}{n[\mu(W-K_\parallel v)^2-c^2(Wv-K_\parallel)^2]}(U_ab_b\delta F^{ab}), \tag{11}$$

where $\mu=(\epsilon+p)/n$. The obtained $\delta n$ and $\delta u$ should be used for the current calculation: $\delta j^a=\Sigma_sq(\delta n_su_s^a+n_s\delta u_s^a)$. We introduce the 4-vector potential $A^a$ such that $\delta F^{ab}=i(k^aA^b-k^bA^a)$ and we apply the Lorentz gauge $k_aA^a=0$. Further calculations are straightforward and after some not lengthy algebra give the dispersion relation

$$D(k)=k_ak^a-(W^2-K_\parallel^2)\times\sum_s\frac{4\pi q_s^2n_s}{\gamma_s^2[\mu_s(W-K_\parallel v_s)^2-c_s^2(Wv_s-K_\parallel)^2]}=0 \qquad (12)$$

for a wave polarized in the $U_a$-$b_a$ plane: $U_aA^a\neq 0$, $b_aA^a\neq 0$. The other dispersion relation, $k_ak^a=0$, describes vacuumlike waves polarized so that $U_aA^a=b_aA^a=0$. The dispersion relation (12) looks exactly like the dispersion relation in the flat space with the substitution $\omega\to W$, $k_\parallel\to K_\parallel$. The metrics $g_{ab}$ is hidden in the definition of $W$ and $K_\parallel$.

We proceed further to the derivation of general relativistic equations for ray propagation given the dispersion equation in the form $D(k,x)$, where the dependence on $x$ includes the (noninertial) effects of rotation. To this end, let us represent a wave packet in the form (see, e.g., Bernstein and Friedland [15])

$$A_b=\int\overline{A_b}(k)\exp(ik_bx^b)\sqrt{|g|}d^4k, \qquad (13)$$

where $k_b$ should satisfy the dispersion relation $D(k,x)=0$. Let us assume that $\overline{A}_b$ has a sharp maximum at $k=k_0$. Then Eq. (13) can be written as follows: $A_b=\exp(ik_{0a}x^a)\int\overline{A_b}(q)\exp(iq_ax^a)\sqrt{|g|}d^4q$, where $q=k-k_0$ is small. Along the ray $x^a=x^a(\lambda)$, the phase $q_ax^a$ should be stationary, so that the equation for the propagation of the maximum is obtained by differentiating the phase with respect to the affine parameter $\lambda$ and equating to zero, which gives $q_a(dx^a/d\lambda)=0$. On the other hand, expanding the dispersion relation near $k_{0a}$, one has $q_a(\partial D/\partial k_a)=0$. Since $q_a$ is arbitrary, one finds

$$\frac{dx^a}{d\lambda}=\frac{\partial D}{\partial k_a}, \qquad (14)$$

where we used the freedom to choose the multiplier for the affine parameter. Using $dD/d\lambda=0$ along the ray, one has $(\partial D/\partial k_a)(dk_a/d\lambda)+(\partial D/\partial x^a)(dx^a/d\lambda)=0$. After substituting Eq. (14), we arrive at the second equation for the ray propagation in the following form:

$$\frac{dk_a}{d\lambda}=-\frac{\partial D}{\partial x^a}. \qquad (15)$$

Affinity (Cristoffel symbols) enters Eq. (15) implicitly via partial derivatives of the terms containing metrics $g_{ab}$. Equation (15) reproduces the ray equation in vacuum $k_bk^a_{;b}=0$ for $D=g^{ab}k_ak_b$. Equations (14) and (15) are the generally covariant generalization of the well-known equations for geometrical optics in a dispersive medium with spatial and temporal inhomogeneity [15]. Similar equations were given by Breuer and Ehlers [9] and Ehlers and Prasanna [13] using Hamiltonian interpretation of $D(k,x)$. The above derivation shows in the most transparent way the relation of the ray equations to the propagation of wave packets. For the functional form of $D$ given in Eq. (12), one has

$$\frac{dx^a}{d\lambda}=2k_a+\frac{\partial D}{\partial W}U^a-\frac{\partial D}{\partial K_\parallel}b^a, \qquad (16)$$

$$\frac{dk^a}{d\lambda}=-D_{,a}-\frac{\partial D}{\partial W}k_cU^c_{,a}+\frac{\partial D}{\partial K_\parallel}k_cb^c_{,a}, \qquad (17)$$

where $D_{,a}=\partial D/\partial x^a$ with $W$ and $K_\parallel$ constant. In Eq. (17), the first term describes the effects of the plasma parameter inhomogeneity, the second term is due to the noninertiality of the frame (it vanishes in the nonrotating observer's frame) or space-time curvature, and the last term is due to the change in the orientation of the magnetic field.

The polarization vector of the wave is determined by the local wave dispersion theory at each point along the ray. Equations for the wave amplitude transport are obtained by expanding the amplitudes $\delta n$, $\delta u$, $A^a$, and the eikonal $S$ in powers of the small parameter $\eta$ (see, e.g., Breuer and Ehlers [9] and Bernstein and Friedland [15]). We do not give them here.

One important example of a system in which the above formalism may be applied is a pulsar magnetosphere. Let $t,r,\phi,z$ be the cylindrical coordinates in the rotating (pulsar) frame, and $t',r',\phi',z'$ be the corresponding coordinates in the nonrotating (observer's) frame. The coordinate transformation is $t'=t$, $r'=r$, $z'=z$, and $\phi'=\phi+\Omega t$. The corresponding metric tensor in the (noninertial) pulsar frame is [14] $g_{tt}=(1-\Omega^2r^2)$, $g_{t\phi}=-\Omega r^2$, $g_{\phi\phi}=-r^2$, and $g_{rr}=g_{zz}=-1$, with the other components equal to zero. We use this metric inside the light cylinder $\Omega r<1$. In the observer's frame, the field pattern is commonly assumed to be rigidly rotating [16] so that all plasma and field parameters depend on $\phi-\Omega t$. As a result, the locally defined wave frequency (in the observer's frame) is not a constant of motion but is a function of $t$. Ray propagation may be treated in the inertial frame, taking into account the changes of the background conditions with $t$. Alternatively, one may treat the propagation in the rotating frame, in which there is inhomogeneity but no time dependence. While the two approaches are physically equivalent, the use of the rotating frame with $\omega=$const along the ray is attractive, albeit at the expense of introducing a nontrivial metric tensor.

In the pulsar frame, $U^a=(g_{tt}^{-1/2},0,0,0)$ is a global timelike velocity field, $U_aU^a=1$. It can be shown that the corresponding $E^a=0$, so that this velocity field satisfies the "magnetized plasma" conditions. Since the metric is time-independent, perturbations have the form $\propto\exp[-i\omega t+iS(r,\phi,z)]$, with $\omega=$const. The local frequencies in the observer's (primed) and pulsar frames are related by $\omega'=\omega-k_\phi\Omega$, which is merely but the Doppler shift. While $\omega$ is constant along the ray since $D$ is time-independent in the pulsar frame, $\omega'$ should change with the change of $k_\phi$. In order to see what might be the effect of the rotation, let us consider the low-frequency, $W^2\ll 4\pi nq^2/\gamma^3$, Alfven wave. In this case, Eq. (12) gives $W^2=K_\parallel^2$ for the Alfven wave, which can be rewritten as $\omega=\mathbf{k}\cdot\mathbf{b}\sqrt{1-\Omega^2r^2}$. Thus, $\partial\omega/\partial\mathbf{r}$ includes a term related to $\partial\sqrt{g_{tt}}/\partial\mathbf{r}$, in addition to those related to the change of the magnetic-field direction **b**. In the observer's frame, this corresponds to a rotating magnetic field, $\mathbf{b}=\mathbf{b}(\mathbf{r},t)$. If $k=(\omega,k_r,k_\phi,0)$, then using Eqs. (16) and (17) one finally finds

$$d\phi/dr=b^\phi/b^r, \quad dt/dr=U^0/b^r, \tag{18}$$

$$dk_r/dr=-[WU^0_{,r}-K_\|(k_r b^r_{,r}+k_\phi b^\phi_{,r})]/K_\| b^r,$$

$$dk_\phi/dr=-(k_r b^r_{,r}+k_\phi b^\phi_{,r})/b^r. \tag{19}$$

Equation (18) shows that the ray follows the field line in the rotating frame. In the nonrotating frame because of the time-dependent transformation, $(d\phi'/dr)=(b^\phi-\Omega U^0_{,r})/b^r$. The last term disappears when $g_{tt}=1$, that is, when $\Omega r$ is neglected. In the equation for $k_r$, the effect of the noninertial frame appears in the additional term $(dk_r/dr)_{\mathrm{add}}=-U^0_{,r}/b^r=-\Omega^2 r/b^r(1-\Omega^2 r^2)^{3/2}$, which may be substantial. A detailed analysis of the ray propagation in the rotating frame will be presented elsewhere.

To conclude, we provide a closed expression for the dispersion relation for low-frequency long-wavelength waves in a relativistic multifluid plasma in a strong magnetic field. We also present a generally covariant form of the equations of geometrical optics for waves in an arbitrary dispersive medium and apply it to the magnetized plasma. We discuss a particular application of the proposed theory to rapidly rotating systems such as pulsars.

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